JOURNAL OF FUNCTIONAL ANALYSIS **158**, 89–112 (1998)
ARTICLE NO. FU973169

Involutive Functionals, Infinite Dimensional Tori, and Neighboring Tori

Martin Schwarz, Jr.

Mathematics Department, Northeastern University, Boston, Massachusetts 02115

Received February 8, 1997; accepted July 9, 1997

In the class of smooth periodic functions, we consider Hamiltonian equations $u_t = K(u)$, constants of the motion $F_m(u)$ that are in involution, and prove an infinite-dimensional version of Liouville's Theorem. We explain in what sense the generic level set of the functionals $F_m(u)$ is an infinite-dimensional torus, why the solution of the Hamiltonian equation is almost periodic in time, and describe how

1

In the class $t \to u(t)$ of smooth periodic functions on $[0, 1]$, we consider Hamiltonian equations

$$\frac{\partial u}{\partial t} = K(u), \tag{1}$$

where $K(u)$ is a nonlinear operator, and also constants of the motion $F_m(u)$ that are in involution which render (1) completely integrable.

In the classical case,

$$\frac{dv}{dt} = K(v), \qquad v \in R^{2N}$$

a theorem of Liouville states that the system is completely integrable. If the involutive constant functions $F_m(v)$, $m = 1, 2, ..., N$ are independent in the sense that their gradients are linearly independent and if the N dimensional level set satisfying $F_m(v) = F_m(v_0)$, $m = 1, 2, ..., N$ is compact; in fact,

(a) the level set is an N dimensional torus on which the flow is quasiperiodic and

(b) neighboring Louville tori are diffeomorphic to one another.

The classical proof of the Liouville theorem is based on the inverse function theorem. For the special case of the Korteweg–de Vries equation,

0022-1236/98 $25.00

McKean and Trubowitz [1] used the method of inverse spectral theory and integrated the equation in the class of smooth periodic functions. They obtained the general solution, and identified the generic invariant set for Korteweg–de Vries with an infinite-dimensional torus, and verified that smooth periodic solutions of KdV are almost periodic in time. Their study [1] did not describe how neighboring tori are related. In previous work [2], we gave a proof of an infinite-dimensional version of statement (a) of Liouville's Theorem, and explained in what sense the generic level set of the functionals $F_m(u)$ is an infinite-dimensional torus and why the solution of (1) is almost periodic in time. An application of [2] to KdV yields a different proof of the result of McKean and Trubowitz. In [2], we were unable to use the implicit function theorem. We introduced instead a local open mapping theorem for certain types of non-differentiable maps. In the present work, we extend the results of [2] to describe how neighboring generic infinite-dimensional tori are related and thus provide a generalization of Liouville's theorem. The proof depends on a variant of the local open mapping theorem introduced in [2]. This approach is related to Lax's [3, 4, 5] study of finite-dimensional level sets of completely integrable partial differential equations and is independent of the method of inverse spectral theory and the viewpoint of algebraic curves. The periodic Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0$$

illustrates the results of the present study.

2

Let H_n $(n \geqslant 0)$ denote the usual Sobolev space of functions on $[0, 1]$, of period one, having derivatives of all orders up to n with norm

$$\|w\|_n^2 = \sum_{j \leqslant n} \int_0^1 |D^j w(x)|^2 \, dx.$$

The norm in the space L_2 is denoted by $\|w\|$. For $w \in H_n$ and integers j, k, and p with $p \geqslant 2$, it is known that

$$\sqrt[p]{\int_0^1 |D^j w(x)|^p \, dx} \leqslant 2^{p-2/2p} \, \|D^k w\|^a \, \|w\|^{1-a},$$

where $a = (j + \frac{1}{2} - 1/p)/k$ and $1 \leqslant j < k \leqslant n$. We denote by C_1^n the space of functions of period one having continuous derivatives of order less than or equal n. The subscript of H_n is generally suppressed.

The Hamiltonian formulation of (1) is due to Lax [3] and is applied to the Korteweg–de Vries, sine-Gordon, nonlinear Schrodinger equations and others. Let $F(u)$ denote an analytic functional whose argument is a smooth functions of period one and let (,) denote the scalar product in L_2. Then

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}(F(u+\varepsilon v)-F(u))=(G_F(u), v)$$

for appropriate u and v defines $G_F(u)$, the gradient of F at u. In studying the different partial differential equations (1), it is convenient to introduce an appropriate Poisson bracket of $F(u)$ with $H(u)$ of the form

$$\{F(u), H(u)\} = (G_F(u), JG_H(u)),$$

where J is an antisymmetric operator independent of u and defined differently for various equations. If $K(u)=JG_F(u)$, then the equation (1) is said to be Hamiltonian. We denote by $S_F(t)\, u$ the nonlinear operator determining the solution of (1) on the basis of its initial values at $t=0 : u(t)=S_F(t)\, u_0$. If $\{F, H\} = 0$ for all u, then the solutions of (1) and of $u_t=JG_H(u)$ commute: $S_H(t)\, S_F(t')=S_F(t')\, S_H(t)$ for all t and t'.

In what follows, let M^n denote the portion of the space H of smooth periodic functions for which n of the functions $F_1, F_2, ...$ have distinct values. In applications, (1) is of the form

$$u_t=[Q, A], \tag{1a}$$

where the operator Q and A depend on u. In particular $Q(u)$ appears as an operator and the spectrum of Q is independent of the nonlinear flow (1). The spectrum $\lambda_m(u)$ are functionals of u that are the integrals of the motion and $F_m(u)$ is a function of $\lambda_m(u)$. For particular equations, additional polynomial type integrals of motion $I_m(u)$ are known explicitly. $I_m(u)$ are directly related to $\lambda_m(u)$. As for KdV,

$$Q=-\frac{d^2}{dx^2}+u$$

and

$$A=4\frac{d^3}{dx^3}-\left(3u\frac{d}{dx}+u_x\right).$$

The eigenvalues $\lambda_m(u)$ of Q are constant along the KdV flow and $F_m(u)=\lambda_m(u)-\lambda_m(0)$. The first three functionals $I_m(u)$ are

$$\int_0^1 u\, dx, \qquad \int_0^1 u^2\, dx, \qquad \int_0^1 \left(\frac{u^3}{6}-\frac{u_x^2}{2}\right) dx.$$

Let $M=\bigcap_{m=1}^{\infty} M^m$. For $u_0 \in M$, we consider the compact set

$$M_{u_0}=\{u \mid F_m(u)=F_m(u_0), m \geqslant 1\}$$

in H. The proof of the theorem uses the fact that $\|u\|$ is constant on M_{u_0} for every u_0 in M. However the statement of the Theorem still holds if we remove this condition and use the properties of the constants $F_m(u)$.

To identify M_{u_0} with the standard infinite-dimensional torus $T^\infty = [0, 1)^\infty$ we first state the result in [2]. Let u be an element of M_{u_0} and view the latter as a subset of L_2. Define $G_m(u)$ to be the gradient of $F_m(u)$ at u. $G_m(u)$ is a vector that is normal to M_{u_0} at u. Let N_u be the closure in L_2 of the span of $G_m(u)$ and assume that $G_m(u)$ is a basis of N_u; by which we mean (a) each element G_u in N_u is uniquely expressible as $G_u = tG(u) = \sum_{m=1}^\infty t_m G_m(u)$ for t in the Hilbert space l_2 and (b) G_u admits the estimate

$$c_1(u)\,|t|_{l_2} \leqslant \|tG(u)\| \leqslant c_2(u)\,|t|_{l_2},$$

where c_1 and c_2 depend continuously on u in M. N_u is the normal space of M_{u_0} at u and, by our assumptions, no single gradient $G_m(u)$ lies in the closure in L_2 of the other gradients $G_n(u)$. The Poisson bracket of $F_m(u)$ and $F_n(u)$ vanishes for all m and n and for u in the class of smooth period one functions. The functionals $F_m(u)$ generate commuting flows

$$\frac{\partial u}{\partial t} = K_m(u) = JG_m(u) \qquad m \geqslant 1 \tag{2}$$

on M_{u_0} and $F_1(u), ..., F_m(u), ...$ are constants of these motions. $K_m(u)$ is tangent to M_{u_0} at u. Denote by T_u the closure in L_2 of the span of $K_m(u)$. Suppose that $K_m(u)$ is a basis of T_u; each element K_u in T_u is uniquely expressible as $K_u = \sum_{m=1}^\infty t_m K_m(u) = tK(u)$ for t in l_2, and

$$c_1(u)\,|t|_{l_2} \leqslant \|tK(u)\| \leqslant c_2(u)\,|t|_{l_2}, \tag{3}$$

where c_1 and c_2 depend continuously on u in M. Assume that T_u equals the orthogonal complement of N_u. T_u represents the tangent space and every direction of L_2 has been accounted for. Let $S_m(t_m)\,u_0$ denote the nonlinear operator uniquely determining the solution of (2) on the basis of its initial values at $t=0: u(t) = S_{F_m}(t_m)\,u_0$. For t in l_2 we show that

$$S(t)\,u = \lim_{N\to\infty} \prod_{m=1}^{N} S_m(t_m)\,u_0$$

in H_n where $S(t+t')\,u = S(t)\,S(t')\,u$ for t, t' in l_2, and for $t \in l_2$ $S(t)\,u \in H_m$ is continuous in t uniformly in u on M_{u_0}. Denote by $dG_F(u)$ the second derivative of F defined by

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(G_F(u+\varepsilon v) - G_F(u)) = dG_F(u)\,v.$$

Let $v(\tau)$, $\tau \geqslant 0$ be a curve in M_{u_0} that satisfies $dv(\tau)/d\tau = K_{v(\tau)}$ with $v(0) = v_1$ and let $dG_{v(\tau)} K_{v(\tau)} = dG_{v(\tau)}/d\tau$ admit the estimate

$$\frac{(dG_u K_v, K'_v)}{\|K_v\| \ \|K'_v\|} \leqslant c \ \|G_v\|, \tag{4}$$

where $K'_v \in T_v$, c is independent of $v(0) \in M_{u_0}$, and $v = v(\tau)$ for small τ. Then $S(t)\, u_0$ is an open map of l_2 onto M_{u_0} in H_n. Let L_{u_0} denote the set of t in l_2 for which $S(t)\, u = u$ for all u in M_{u_0}. S is a homeomorphism of l_2/L_{u_0} onto M_{u_0} in H_n. l_2/L_{u_0} is compact and may be identified as in [2] with the standard infinite-dimensional torus T^∞: in more detail, there exist ω_m, $m \geqslant 1$ from L_{u_0} so that each t of l_2/L_{u_0} is uniquely represented by $t = \sum_{m=1}^{\infty} \tau_m \omega_m$ where $0 \leqslant \tau_m < 1$ for all m. M_{u_0} is an infinite-dimensional torus and the solution $S_m(t_m)\, u$ is almost periodic on l_2/L_{u_0}, uniformly with respect to initial values $u \in M_{u_0}$.

The identification of M_{u_0} with an infinite-dimensional torus and the proof of the almost periodicity of the flow of $S_m(t_m)\, u_0$ depend upon the following assumptions: (a) the compactness of M_{u_0}, (b) $\{F_m(u), F_n(u)\} = 0$ for all m and n, (c) the assumptions that the sequences K_m and G_m are a bases for T and N, respectively, with $N \oplus T = L_2$, and (d) the estimate of dG. These assumptions are analogous to the conditions imposed in the finite-dimensional Liouville Theorem. Examples are known which show that our arguments are not valid unless something like the assumption (d) is satisfied, specifically that the gradients G_m satisfy a growth condition.

In the present study, it is supposed that $\sum_{m=1}^{\infty} F_m^2(u)$ is bounded uniformly in u on bounded sets in H. Let $F(u) = (F_1(u), ..., F_m(u), ...)$ and $l = F(M) \in l_2$. In what follows, we use the estimate (3) of K_u and the corresponding estimate of G_u where the constants $c_1(u)$ and $c_2(u)$ are locally independent of u in a small neighborhood of any u_0 in M.

Let $v_0 \in M$ and $f_0 = F(v_0) \in l$. We write u_{f_0} for v_0. For different f in a small neighborhood of f_0 in l, we pick out distinct initial points u_f close to u_{f_0} in M with $f = F(u_f)$. In particular, we prove that for f in a small neighborhood of f_0, there exist u_f in M with $f = F(u_f)$ and

$$|f - f_0|_{l_2} \geqslant c \ \|u_f - u_{f_0}\|,$$

where c is locally independent of $u \in M$ and $f \in l$. This result will be obtained by constructing a curve $u(f, s)$, $0 \leqslant s \leqslant 1$ that is continuous in H, connects u_{f_0} with u_f in M, and remains in M except for a countable number of values of s. The curve is relatively short in the sense that the length of the curve in H joining u_{f_0} with u_f is bounded by a fixed multiple of

$|f-f_0|_{l_2}$. The curve that joins u_{f_0} to u_f depends uniquely on u_{f_0}. The construction begins with u_{f_0} and the direction

$$(f-f_0)\, G_{u_{f_0}} = \sum_{m=1}^{\infty} (f_m - f_m^0)\, G_m(u_{f_0})$$

transverse to $M_{u_{f_0}}$ at u_{f_0}. For u in M, use the analyticity of $F_m(u)$ to verify that, locally, any line in H through u is contained in M with the exception of a countable number of elements. If

$$\left\| \frac{\partial u}{\partial \tau} - (f-f_0)\, G_{u_{f_0}} \right\| \leqslant \alpha_0\, \|(f-f_0)\, G_{u_{f_0}}\|$$

at $\tau=0$, α_0 being a small fixed constant, then there is a curve $u(\tau)$ for τ small enough that remains in M except for a countable number of elements and $|f-F(u(\tau))|_{l_2}$ is less than $|f-f_0|_{l_2}$ by an amount that is comparable to the angle between

$$(f-f_0)\, G_{u_{f_0}} \qquad \text{and} \qquad \frac{\partial u}{\partial \tau}$$

at $\tau=0$. If for $u \in M$ and directions v transverse to M_u at u, $dG_m(u)\, v$ admits the estimate

$$\|dG_m(u)\, v\| \leqslant c_m\, \|v\|,$$

where c_m is square summable locally uniformly in u and v in M, then iteration leads to u_f in M with $f=F(u_f)$ and

$$|f-f_0|_{l_2} \geqslant c\, \|u_f - u_{f_0}\|,$$

where c is locally independent of $u \in M$ and $f \in l$.

Next, we prove that the torus $M_{u_f} = F^{-1}(f)$ is homeomorphic to the standard torus T^∞. We establish that $M_{u_{f_0}}$ is characterized by basic generators $\omega_m(f_0)$ that are periods of $S(t)\, u_{f_0}$, and that for f in a small enough neighborhood of f_0 in l, the $\omega_m(f_0)$ may be continuously extended to the basic generators $\omega_m(u_f)$ that describe M_{u_f}. This gives a relatively short curve continuous in H and contained in M except for a countable number of elements.

The inverse image of $F(u)=f_0$ equals $M_{u_{f_0}}$ and $S(t)\, u_{f_0}$ is a homeomorphism of l_2/L_{f_0} onto $M_{u_{f_0}}$. We first identify l_2/L_{f_0} with the set T_{f_0} of convergent sums

$$\sum_{m=1}^{\infty} \tau_m \omega_m(f_0), \qquad 0 \leqslant \tau_m < 1,$$

where the convergence is in l_2, and $\omega_m(f_0) \in L_{f_0}$ is a complete minimal l_2 sequence that generates L_{f_0}: the closure in l_2 of the integral span of ω_m equals $L_{u_{f_0}}$. For f in a small enough neighborhood of f_0 in l, we find $\omega_m(f) \in L_f$ with

$$|\omega_m(f) - \omega_m(f_0)|_{l_2} \leqslant \tfrac{1}{2} |\omega_m(f_0)|_{l_2},$$

where $\omega_m(f)$ converges to $\omega_m(f_0)$ in l_2 as f tends to f_0 in l. Furthermore, $\omega_m(f)$ is a complete minimal l_2 sequence that generates L_f, and $M_{u_f} = F^{-1}(f)$ is identified with the set T_f of convergent sums $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$, $0 \leqslant \tau_m < 1$, with convergence in l_2 uniformly in τ_m and f. Furthermore, there exists a curve $\sum_{m=1}^{\infty} \tau_m \omega_m(f, s)$, $0 \leqslant \tau_m < 1$, continuous in l_2 that connects $\sum_{m=1}^{\infty} \tau_m \omega_m(f_0)$ with $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$ and is relatively short in the sense that the length of the curve in l_2 is less than $\frac{1}{2} |\sum_{m=1}^{\infty} \tau_m \omega_m(f_0)|_{l_2}$. T_f is homeomorphic to T_{f_0}, and for f in a small enough neighborhood of f_0 in l, T_f is uniformly close to T_{f_0}. We verify that $S(t)\,u$ in H is continuous in (t, u) from $l_2 \times M$. This leads to a result which states a sense in which M_f is connected with $M_{u_{f_0}}$.

THEOREM. *Let $F_m(u)$ be a sequence of analytic functions of $u \in H$ that are in involution and square summable uniformly in u on bounded sets in H. For u in M, M_u is compact in H, and suppose that $K_m(u)$ and $G_m(u)$ are a basis for T_u and N_u respectively with $N_u \oplus T_u = L_2$. If, for u in M and directions v transverse to M_u at u, $dG_m(u)$ admits the estimate $\|dG_m(u)\,v\| \leqslant c_m \|v\|$ where c_m is square summable locally uniformly in u and v, then for f in a small neighborhood of $f_0 \in l$ there exists $u_f \in M$ satisfying $F(u_f) = f$ that admits the estimate $|f - f_0|_{l_2} \geqslant c\,\|u_f - u_{f_0}\|$ where $c \geqslant 0$ is locally independent of u in M and f in l. Let dG satisfy the estimate (4). Then M_{u_f} is homeomorphic to $M_{u_{f_0}}$ and to the standard infinite-dimensional torus T^{∞}. Furthermore M_{u_f} and M_{u_f} are connected by a relatively short continuous curve in H that is contained in M except for a countable number of elements.*

In Section 2 we prove the theorem and in Section 3 we apply this result to Korteweg-de Vries. A discussion of the proof was presented at the AMS summer school at Amherst in 1990.

3

In this section, we prove the theorem. Lemma 1 shows that for f in a small neighborhood of $f_0 \in l$ there exists $u_f \in M$ near u_{f_0} where $f = F(u_f)$. The elements u_f of M correspond to distinct tori M_{u_f} and are connected with u_{f_0} in M by a relatively short curve. Lemma 2 establishes the sense in which M_{u_f} is connected by a continuous curve in H to $M_{u_{f_0}}$.

LEMMA 1. *For $f_0 = F(u_{f_0}) \in l$, there is a number δ such that if $f \in l$ and $|f - f_0|_{l_2} \leqslant \delta$, then there exits a curve $u(f, s)$, $0 \leqslant s \leqslant 1$ in H that is contained in M except for a countable number of values of s and connects the initial value u_{f_0} with $u(f, 1) = u_f \in M$; $f = F(u_f)$ admits the estimate*

$$c\,|f - f_0|_{l_2} \geqslant \|u_f - u_{f_0}\|$$

where c is locally independent of u in M and $f \in l$. The curve is relatively short in the sense that the length of the curve in H joining u_{f_0} with u_f is bounded by a fixed multiple of $|f - f_0|_{l_2}$. The function $u(f, s)$ in H_n is continuous in s uniformly in f and continuous in f uniformly in s. Furthermore, $u(f, s)$ converges in H_n to u_{f_0} uniformly in s as f tends to f_0 in l.

Proof. For $f \in l$ and $f_0 = F(u_{f_0})$, we construct a curve $u(\tau)$, $\tau \geqslant 0$ in H starting at $u(0) = u_{f_0}$ and ends at u with $F(u) = f$, and is contained in M except for an countable number of values τ. We apply the estimate, $c_1\,|t|_{l_2} \leqslant \|tG_u\| \leqslant c_2\,|t|_{l_2}$ where c_1 and c_2 are independent of u in a small neighborhood of u_{f_0} in M. We select δ to keep u in this neighborhood of u_f in M.

Let $\delta_1 = \delta/2$ and $f \in l$ satisfy $\delta_1 \leqslant |f - f_0|_{l_2} \leqslant \delta$. The subscript l_2 in the norm is suppressed throughout this section. Begin with u_{f_0} and the direction $(f - f_0)\,G_{u_{f_0}}$ that is transverse to $M_{u_{f_0}}$ at u_{f_0} in M; and locally, any line through u_{f_0} in H is contained in M with the exception of an countable number of elements. For directions $R(u)$ near $(f - f_0)\,G_{u_{f_0}}$, there exists a local flow $du/d\tau = R(u)$, $u(\tau)$ is bounded in H_m locally in τ, $u(0) = u_{f_0}$, and $du(0)/d\tau = \dot{u}(0)$ satisfies

$$\|\dot{u}(0) - (f - f_0)\,G_{u_{f_0}}\| \leqslant \alpha_0\,\|(f - f_0)\,G_{u_{f_0}}\|,$$

where $\alpha_0 \ll 1$ is an absolute constant. It can be shown that

$$\left\{\int_0^\tau \int_0^1 (\dot{u}(x, s) - \dot{u}(x, 0))^2\,dx\,ds\right\}^{1/2} \leqslant \sqrt{\tau}\,\alpha_0\,\|\dot{u}(0)\| \tag{5}$$

locally in τ for $0 \leqslant \tau < \tau_1(\delta_1)$. Use (5) to check that

$$\frac{((f - f_0)\,G_{u_0}, \dot{u}(0))}{\|(f - f_0)\,G_{u_0}\|\,\|\dot{u}(0)\|} \geqslant \frac{1 - \alpha_0}{1 + \alpha_0} = \alpha. \tag{6}$$

Next expand $|f - F(u(\tau))|^2$ and divide by $|f - f_0|^2$ to obtain

$$\frac{|f - F(u(\tau))|^2}{|f - f_0|^2} = 1 - 2\sum_m \frac{(f_m - F_m(u_{f_0}))(F_m(u(\tau)) - F_m(u_{f_0}))}{|f - f_0|\,\|\dot{u}(0)\|}\,\frac{\|\dot{u}(0)\|}{|f - f_0|} + \frac{|F(u(\tau)) - F(u_{f_0})|^2}{|f - f_0|^2}.$$

We establish the estimate

$$\sum_{m=1}^{\infty} (f_m - f_m^0)(F_m(u(\tau)) - F_m(u_{f_0})) \geqslant \frac{\tau}{2} c_1 \alpha \, |f - f_0| \, \|\dot{u}(0)\|. \tag{6a}$$

F_m is twice Frechet differentiable and

$$\sum_{m=1}^{N} (f_m - f_m^0)(F_m((u(\tau)) - F_m(u_{f_0}))$$

equals

$$\int_0^\tau \sum_{m=1}^{N} ((f_m - f_m^0)(G_m(u(s)) - G_m(u_{f_0})), \dot{u}(s)) \, ds$$
$$+ \int_0^\tau \left(\sum_{m=1}^{N} (f_m - f_m^0) \, G_m(u_{f_0}), \dot{u}(s) \right) ds.$$

To estimate the first term, use the fact that $u(\tau)$ is contained in M except for a countable number of τ's, the estimates for dG_m, the estimate (5), and proceed as follows:

$$\begin{aligned}
&\left| \int_0^\tau \left(\sum_{m=1}^{N} (f_m - f_m^0)(G_m(u(s)) - G_m(u_0)), \dot{u}(s) \right) ds \right| \\
&\quad = \left| \int_0^\tau \int_0^s \left(\sum_{m=1}^{N} (f_m - f_m^0) \, dG_m(u(\eta)) \, \dot{u}(\eta), \dot{u}(s) \right) d\eta \, ds \right| \\
&\quad \leqslant \int_0^\tau \int_0^s \left\| \sum_{m=1}^{N} (f_m - f_m^0) \, dG_m(u(\eta)) \, \dot{u}(\eta) \right\| \|\dot{u}(s)\| \, d\eta \, ds \\
&\quad \leqslant c \left\{ \sum_{m=1}^{N} (f_m - f_m^0)^2 \right\}^{1/2} \int_0^\tau \left\{ s \int_0^s \|\dot{u}(\eta)\|^2 \, d\eta \right\}^{1/2} \|\dot{u}(s)\| \, ds \\
&\quad \leqslant c \left\{ \sum_{m=1}^{N} (f_m - f_m^0)^2 \right\}^{1/2} \|\dot{u}(0)\|^2 \, \tau^2.
\end{aligned}$$

Rewrite the second term, use (5) in the form $\|u(\tau) - u(0) - \tau \dot{u}(0)\| \leqslant \tau \alpha_0 \, \|\dot{u}(0)\|$, and estimate as follows:

$$\begin{aligned}
&\left| \int_0^\tau \left(\sum_{m=1}^{N} (f_m - f_m^0) \, G_m(u_0), \dot{u}(s) \right) ds \right| \\
&= \left| \tau \left(\sum_{m=1}^{N} (f_m - f_m^0) \, G_m(u_0), \dot{u}(0) \right) + \left(\sum_{m=1}^{N} (f_m - f_m^0), u(\tau) - u(0) - \tau \dot{u}(0) \right) \right| \\
&\geqslant \tau(\alpha - \alpha_0) \left\| \sum_{m=1}^{N} (f_m - f_m^0) \, G_m(u_0) \right\| \|\dot{u}(0)\|.
\end{aligned}$$

Combine, let N tend to infinity, use the fact that $\|\dot{u}(0)\|$ is comparable to $|f-f_0|$ and select τ small depending on δ_1 to obtain (6a).

Now

$$\frac{|f-F(u(\tau))|^2}{|f-f_0|^2} \leqslant 1 - \frac{\tau c_1 \alpha}{4} \frac{\|\dot{u}(0)\|}{|f-f_0|} + \frac{|F(u(\tau)) - F(u_0)|^2}{|f-f_0|^2}. \tag{7}$$

To bound $|F(u(\tau)) - F_m(u_0)|^2/|f-f_0|^2$ we first obtain the estimate

$$\|F(u(\tau)) - F(u_0)\|^2 \leqslant c\tau^2 \, \|\dot{u}(0)\| \, |f-f_0|.$$

Use the fact that $F_m(u)$ is Frechet differentiable, (5), and that $\|\dot{u}(0)\|$ is comparable to $|f-f_0|$ to obtain the estimate

$$\begin{aligned}
&\sum_{m=1}^{N} (F_m(u(\tau)) - F_m(u_0))^2 \\
&\quad = \int_0^\tau \left(\sum_{m=1}^{N} (F_m(u(\tau)) - F_m(u(0)))\, G_m(u(s)), \dot{u}(s) \right) ds \\
&\quad \leqslant \int_0^\tau \left\| \sum_{m=1}^{N} (F_m(u(\tau)) - F_m(u_0))\, G_m(u(s)) \right\| \, \|\dot{u}(s)\| \, ds \\
&\quad \leqslant c_2 \left\{ \sum_{m=1}^{N} (F_m(u(\tau)) - F_m(u_0))^2 \right\}^{1/2} \int_0^\tau \|\dot{u}(s)\| \, ds \\
&\quad \leqslant c_2 \tau \, \|\dot{u}(0)\| \left\{ \sum_{m=1}^{N} (F_m(u(\tau)) - F_m(u_0))^2 \right\}^{1/2}
\end{aligned}$$

which leads to the result. Adjust τ to obtain

$$\frac{|F(u(\tau)) - F(u_0)|^2}{|f-f_0|^2} \leqslant \frac{\tau c_1 \alpha}{8} \frac{\|\dot{u}(0)\|}{|f-f_0|}.$$

Combine with (7) to find

$$\frac{|f-F(u(\tau))|^2}{|f-f_0|^2} \leqslant 1 - \frac{\tau c_1 \alpha}{8} \frac{\|\dot{u}(0)\|}{|f-f_0|}.$$

Take the square root of the inequality, apply Cauchy's inequality, and then divide by $|f-f_0|$ to obtain

$$|f-F(u(\tau))| \leqslant |f-f_0| - \frac{\tau c_1 \alpha}{16} \|\dot{u}(0)\|.$$

Similarly,

$$|f - F(u(\tau))| \leqslant |f - f_0| - c\alpha \, \|u(\tau) - u(0)\|.$$

Iteration gives u_n in M that satisfies

$$|f - F(u_n)| \leqslant |f - F(u_{n-1})| - c\tau \, \|\dot{u}_{n-1}(0)\| \tag{8}$$

and

$$|f - F(u_n)| \leqslant |f - F(u_{n-1})| - c\alpha \, \|u_n - u_{n-1}\|$$

where c is an absolute constant independent of τ and δ_1. Combine (8) and use that $\|\dot{u}_{n-1}(0)\|$ is comparable to $|f - F(u_{n-1})|$ to verify

$$|f - F(u_n)| \leqslant (1 - c\tau)^n \, |f - F(u_{f_0})| \leqslant (1 - c\tau)^n \, \delta.$$

Select n so that $|f - F(u_n)| \leqslant \delta_1$. Combine and find u_m in M where the norm of u_m in M is bounded independently of m and admits the estimate

$$|F(u_m) - f| < \frac{\delta}{2^m}$$

and

$$c \, \|u_m - u_{m'}\| \leqslant |F(u_m) - F(u_{m'})|.$$

It follows that u_m converges to u_f in M and that $F(u_f) = f$ satisfies

$$c \, \|u_f - u_{f_0}\| \leqslant |f - F(u_{f_0})|,$$

where c is an absolute constant. The result follows.

This completes the proof of Lemma 1.

The inverse image of $F(u) = f_0$ equals M_{f_0}, and $S(t)\, u_{f_0}$ identifies M_{f_0} with l_2/L_{f_0}. Following [2], l_2/L_{f_0} is identified with a compact subset of l_2 in which each element is uniquely represented by $\sum_{m=1}^{\infty} \tau_m \omega_m$, $0 \leqslant \tau_m < 1$ in which $\omega_m \in L_{f_0}$ is a complete minimal l_2 sequence. The sense in which M_{u_f} is homeomorphic to $M_{u_{f_0}}$ is established in Lemma 2.

LEMMA 2. *Let $f_0 = F(u_{f_0})$ in l. For f in a small enough neighborhood of $f_0 \in l$ there exists a complete minimal l_2 sequence $\omega_m(f)$ such that $\omega_m(f)$ converges to ω_m in l_2 as f approaches $f_0 \in l$. Each element of l_2/L_f is identified with $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$ in l_2 for some τ_m, $0 \leqslant \tau_m < 1$ where the convergence is in l_2, uniformly in τ_m and locally uniformly in f. Furthermore, there exists a curve $\sum_{m=1}^{\infty} \tau_m \omega_m(f, s)$, $0 \leqslant s \leqslant 1$ in l_2 that is continuous in s, connects $\sum_{m=1}^{\infty} \tau_m \omega_m(f_0)$ with $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$, and is relatively short in the* *sense that the length of the curve in* l_2 *is less than* $\frac{1}{2}|\sum_{m=1}^{\infty}\tau_m\omega_m(f_0)|_{l_2}$. *In addition,* $\sum_{m=1}^{\infty}\tau_m\omega_m(f,s)$ *converges in* l_2 *uniformly in* f, s, *and* τ_m, *and in* l_2, $\sum_{m=1}^{\infty}\tau_m\omega_m(f,s)$ *is continuous in* s *uniformly in* τ_m *and* f, *and continuous in* f *uniformly in* s *and* τ_m.

Proof. We first construct a complete minimal l_2 sequence $\omega_m(f)\in L_{u_f}$ so that $\omega_m(f)\to\omega_m$ in l_2 as f tends to $f_0\in l$.

Step 1. Following the proof of Lemma 3 in [2], there exists $\omega_m\in L_{u_{f_0}}$ such that ω_m is a complete minimal l_2 sequence that generates $L_{u_{f_0}}$: the closure in l_2 of the integral span of ω_m equals $L_{u_{f_0}}$. Use the compactness of $M_{u_{f_0}}$ and that S is an open map to verify that $|\sum_{m=1}^{n}\omega_m|_{l_2}$ with all rearrangements of the sum is bounded independently of n. It follows that $\sum_{m=1}^{\infty}\tau_m\omega_m$, $0\leqslant\tau_m<1$ converges in l_2 uniformly in τ_m, and that $\sum_{m=1}^{\infty}|\omega_m|_{l_2}^2$ is finite. It can be shown that the set of all convergent sums $\sum_{m=1}^{\infty}\tau_m\omega_m$, $0\leqslant\tau_m<1$ with identification of $\tau_m=1$ and $\tau_m=0$ is a compact subset of l_2.

The construction of $\omega_m(f)$ depends on the following estimate. Let $v_f=S(\omega_m)\,u_f$. For f in a small enough neighborhood of $f_0\in l$ there is a number $\delta>0$ independent of m such that $\|u_f-v_f\|\leqslant\delta$. The estimate of $u_f-v_f=u_f-S(\omega_m)\,u_f$ is developed as follows. Let $\omega_m^n=(\omega_m^1,...,\omega_m^n,0,...)$ denote the truncation of ω_m in l_2. Then

$$\begin{aligned}\|u_f-v_f\| &\leqslant \|u_f-u_{f_0}\|+\|S(\omega_m)\,u_{f_0}-S(\omega_m)\,u_f\|\\ &\leqslant \|u_f-v_{f_0}\|+\|S(\omega_m)\,u_{f_0}-S(\omega_m^n)\,u_{f_0}\|\\ &\quad+\|S(\omega_m^n)\,u_{f_0}-S(\omega_m^n)\,u_f\|+\|S(\omega_m^n)\,u_f-S(\omega_m)\,u_f\|\\ &\leqslant c\,|f-f_0|_{l_2}+2c_2\,|\omega_m-\omega_m^n|_{l_2}+\|S(\omega_m^n)\,u_{f_0}-S(\omega_m^n)\,u_f\|,\end{aligned}$$

where we have used the estimate in Lemma 1 and then (3) to verify that $\|S(t)\,u-S(t')\,v\|\leqslant c\,|t-t'|_{l_2}$ where c is independent of u and v from bounded sets in M. For m large, consider the estimation

$$\|u_f-v_f\|\leqslant\|S(\omega_m)\,u_f-u_f\|\leqslant c_2\,|\omega_m|_{l_2}.$$

Use that $\omega_m\to 0$ in l_2 as $m\to\infty$ to select $|\omega_m|_{l_2}$ small for m large enough. For finite m, use the convergence of $\omega_m^n(f_0)$ to $\omega_m(f_0)$ to select for n large and that $S(t)\,u$ in L_2 is continuous in u from M to obtain the estimate.

Next we construct $\omega_m(f)$ from ω_m. By construction u_f and $v_f=S(\omega_m)\,u_f$ are elements of the same level set M_{u_f}. By modifying a technique introduced in [2], we prove that for real numbers $\kappa_m\ll 1$ and $S(\omega_m)\,u_f$ close enough to u_f independently of m, there exist $w_m = \omega(f)_m - \omega_m$ with $S(w_m)\, v_f = u_f$ that satisfy

$$\|u_f - v_f\|\, \kappa_m \geqslant c_1\, |\omega_m(f) - \omega_m|_{l_2},$$

where c_1 is an absolute constant.

The subscript f is dropped and the subscript l_2 in the notation for the norm in the space l_2 is suppressed in the remaining part of Step 1. Let $K_v = P_{T_v}(u-v)$ denote the projection of $u-v$ onto T_v. In what follows we consider the evolution $\partial v/\partial\tau = K_v$, $\tau \geqslant 0$ with $v(0) = v$. Use of (3), and the fact that $K_m(v) \in L_2$ is continuous in $v \in M$ confirms that K_v is continuous in $v \in M$. Since $S(t)\, v$, $t \in l_2$ in H_m is defined for each v in M_{u_f}, the solution $v(\tau)$, $\tau \geqslant 0$ in M_{u_f} of the initial value problem $\partial v/\partial\tau = K_v$, $\tau \geqslant 0$ is defined. We construct a curve joining v with u in M_{u_f}. This depends initially on $(K_v, u-v)/\|K_v\|\,\|u-v\|$ and the sense in which the angle changes along $v(\tau)$, $\tau \geqslant 0$. Since $K_v = P_{T_v}(u-v)$ may not determine a curve that brings you closer to u we adjust the direction. This gives a local solution which initially may not be closer to u but finally reaches an element in M_{u_f} closer to u. This construction uses the fact that $\|u\|$ is constant on M_f. However, this condition can be replaced with an infinite or finite codimension level set comprised of $F_m(u)$. If $u - v \in N_v$, there exists $dF_v \in N_v$ and $u - v = dF_v = \sum_m c_m\, dF_m(v)$. Consider the portion of M with $\sum_m c_m(F_m(v) - F_m(0))$ constant, and translate T_v along $u-v$ as $u_\eta = v + \eta(u-v) + t_\eta K(v)$. Use that $\sum_m c_m(F_m(u_\eta) - F_m(0)) = c$ independently of η, and standard estimates to find estimates comparable to the estimates in intems 2–4. Quantitative estimates used in this construction are developed in the next five items.

Item 1. The sense in which $(K_v, u-v)/\|K_v\|\,\|u-v\|$ varies along the solution $\partial u/\partial\tau = K_v$ is determined by

$$\frac{d}{d\tau}\frac{(u-v(\tau), K_{v(\tau)})}{\|u-v(\tau)\|\;\|K_{v(\tau)}\|} = \frac{-\|K_{v(\tau)}\|}{\|u-v(\tau)\|}\left\{1 - \frac{\|K_{v(\tau)}\|^2}{\|u-v(\tau)\|^2} + \frac{(dG_{v(\tau)}K_{v(\tau)}, K_{v(\tau)})}{\|K_{v(\tau)}\|^2}\right\}.$$

Begin with $(u-v(\tau), K_{v(\tau)})/\|u-v(\tau)\|\;\|K_{v(\tau)}\|$, differentiate in τ, and then use $dv/d\tau = K_v$ and $u - v = K_v + G_v$ to find

$$\frac{d}{d\tau}\frac{(u-v(\tau), K_{v(\tau)})}{\|u-v(\tau)\|\;\|K_{v(\tau)}\|} = \frac{-\|K_{v(\tau)}\|}{\|u-v(\tau)\|} + \frac{\|K_{v(\tau)}\|^3}{\|u-v(\tau)\|^3} + \frac{(G_{v(\tau)}, dK_{v(\tau)}K_{v(\tau)})}{\|K_{v(\tau)}\|\;\|u-v(\tau)\|}.$$

Differentiate $(G_{v(\tau)}, K_{v(\tau)}) = 0$ with respect to τ and then substitute

$$(dG_{v(\tau)}K_{v(\tau)}, K_{v(\tau)}) = -(G_{v(\tau)}, dK_{v(\tau)}K_{v(\tau)})$$

into the proceeding identity to obtain the result.

Item 2. Fix $0 \leqslant \eta \leqslant 1$. For $v \in M_{u_f}$ there is a function u_η in L_2 such that $\|u_\eta\| = \|u_f\|$, $K_v = T_v(u_\eta - v)$ satisfies

$$\frac{(K_v, u_\eta - v)}{\|K_v\| \ \|u_\eta - v\|} \geqslant 1 - \sqrt{\eta},$$

and $\|u_\eta - v\| = \sqrt{\eta}\ \|u - v\|$.

The elements u and v belong to M_{u_f} and satisfy $\|u\| = \|v\| = \|u_f\|$. Therefore u and v are elements of N_u and N_v, respectively. Translate T_v along $u - v$ and find $K(v) \in T_v$ so that

$$u_\eta = v + \eta(u - v) + K(v)$$

satisfies $\|u_\eta\| = \|u_f\|$ and $\|K(v)\| \geqslant (\sqrt{\eta} - \eta)\ \|u - v\|$. If $K(v)$ is in the direction of $P_{T_v}(u - v)$, then a direct calculation shows that $\|u_\eta - v\|^2 = \eta\ \|u - v\|^2$ and $\|u_\eta - u\|^2 = (1 - \eta)\ \|u - v\|^2$. Use the estimates for $\|K(v)\|$ and $\|u - v\|$ to estimate the angle between $u_\eta - v$ and the projection of $u_\eta - v$ onto T_v.

Item 3. In this step $v(\tau)$, $\tau \geqslant 0$ corresponds to the solution in M_{u_f} determined by $K_v = P_{T_v}(u_\eta - v)$. We establish

$$\|u_\eta - v(\tau)\| \leqslant \|u_\eta - v\| - \frac{1}{4} \frac{(K_v, u_\eta - v)}{\|K_v\| \ \|u_\eta - v\|} \|u_\eta - v\| \qquad \text{for} \quad \tau \leqslant c_0 \frac{(K_v, u_\eta - v)}{\|K_v\| \ \|u_\eta - v\|},$$

where c_0 is an absolute constant and

$$\|v(\tau) - v\| \geqslant \frac{\tau}{2} \|K_v\|.$$

This will depend on the estimate

$$\|dK_{v(\tau)} K_{v(\tau)}\| \leqslant c\ \|K_{v(\tau)}\|,$$

where c is an absolute constant. Use that $N_{v(\tau)} \oplus T_{v(\tau)} = L_2$ and write $dK_{v(\tau)} K_{v(\tau)} = K'_{v(\tau)} + G'_{v(\tau)}$. Differentiate $u_\eta - v(\tau) = K_{v(\tau)} + G_{v(\tau)}$ in τ, multiply the resulting expression by $K'_{v(\tau)} + G'_{v(\tau)}$, and integrate in x from zero to one. Then differentiate $(K_{v(\tau)}, G_{v(\tau)}) = 0$ in τ, and substitute $(dK_{v(\tau)} K_{v(\tau)}, G'_{v(\tau)}) = -(dG'_{v(\tau)} K_{v(\tau)}, K_{v(\tau)})$ into the proceeding identity, and use the estimate for $dG_{v(\tau)}$ to establish the result.

Now

$$v(\tau) - v - \tau K_v = \int_0^\tau \left\{ \int_0^s \frac{d}{ds'} K_{v(s')}\, ds' \right\} ds. \tag{9}$$

Estimate in L_2 and use the bound for $\|dK_{v(\tau)}K_{v(\tau)}\|$ and that $\|K_{v(\tau)}\|/\|u_\eta - v(\tau)\| \leqslant \|K_v\| \; \|u-v\|$ from item one to prove

$$\|v(\tau) - v - \tau K_v\| \leqslant \left\{\int_0^\tau \int_0^s \int_0^1 dK_{v(s')} K_{v(s')} \, dx \, ds' \, ds\right\}^{1/2} \leqslant \tau^2 c \; \|K_v\|. \tag{10}$$

If $c\tau \leqslant 1/2$

$$\|v(\tau) - v\| \geqslant \frac{\tau}{2} \, \|K_v\|.$$

Estimates (9) and (10) are used to establish

$$\frac{(u_\eta - v(\tau), v(\tau) - v)}{\|u_\eta - v(\tau)\| \; \|v(\tau) - v\|} \geqslant \frac{1}{2} \frac{(K_v, u_\eta - v)}{\|K_v\| \; \|u_\eta - v\|}, \tag{11}$$

where τ is bounded above by a fixed multiple of $(K_v, u_\eta - v)/\|K_v\| \; \|u_\eta - v\|$. Next expand $\|u_\eta - v\|^2 = \|u_\eta - v + v - v(\tau)\|^2$ and divide by $\|u_\eta - v\|^2$ to obtain

$$\frac{\|u_\eta - v(\tau)\|^2}{\|u_\eta - v\|^2} = 1 - 2 \frac{(u_\eta - v, v(\tau) - v)}{\|u_\eta - v\| \; \|v(\tau) - v\|} \frac{\|v(\tau)\|}{\|u_\eta - v\|^2}.$$

Apply (11) and (9) and adjust the proceeding τ and find

$$\begin{aligned} \frac{\|u_\eta - v(\tau)\|^2}{\|u_\eta - v\|^2} &\leqslant 1 - \frac{(K_v, u_\eta - v)}{\|K_v\| \; \|u_\eta - v\|} \frac{\|v(\tau) - v\|}{\|u_\eta - v\|} + \frac{\|v(\tau) - v\|^2}{\|u_\eta - v\|^2} \\ &\leqslant 1 - \frac{1}{2} \frac{(K_v, u_\eta - v)}{\|K_v\| \; \|u_\eta - v\|} \frac{\|v(\tau) - v\|}{\|u_\eta - v\|}. \end{aligned}$$

Take the square root of the inequality, apply Cauchy's inequality, and then divide by $\|u_\eta - v\|$ to obtain the result.

Item 4. Begin with u_η as in items 3 and 4. Fix $0 \leqslant \kappa \leqslant 1$. If $v \in M_{u_f}$ with $\|u - v\|$ is small enough number δ and if η is chosen small depending on κ then there is a $t \in l_2$ and $S(t)\, v \in M_{u_f}$ that satisfy

$$\|S(t)\, v - u_\eta\| \leqslant \frac{\kappa}{2} \sqrt{\eta} \; \|u_\eta - v\| \tag{12}$$

and

$$c_1 \; |t|_{l_2} \leqslant \|S(t)\, v - u_\eta\|, \tag{13}$$

where c_1 is independent of v and f.

The projection of $u-v$ onto T_v determines a curve in M_{u_f}. Since this vector may not determine a curve that brings you closer to u we adjust the direction. To determine the direction we translate T_v along $u-v$ and use Item 2 to find $u_\eta \in L_2$ with $\|u_\eta\| = \|u_f\|$. Now $K_v = P_{T_v}(u_\eta - v)$ satisfies

$$\frac{(K_v, u_\eta - v)}{\|K_v\| \; \|u_\eta - v\|} \geqslant 1 - \sqrt{\eta}.$$

Let $v(\tau)$ denote the curve determined by $K_v = P_{T_v}(u_\eta - v)$ and let

$$\alpha(\tau, \eta) = \frac{(K_{v(\tau)}, u_\eta - v)}{\|K_{v(\tau)}\| \; \|u_\eta - v\|}.$$

Use Item 1, the estimate of $(dG_v K_v, K_v)$ and the fact that $\|G_{v(\tau)}\| \leqslant \|u_\eta - v(\tau)\| \leqslant \|u_\eta - v\| = \sqrt{\eta}\, \|u - v\| \leqslant \delta \sqrt{\eta}$ to show that $\alpha(\tau, \eta)$ satisfies

$$-\alpha + \alpha^3 - \sqrt{\eta}\, c\, \delta\alpha \leqslant \frac{d\alpha}{d\tau} \leqslant -\alpha + \alpha^3 + \sqrt{\eta}\, c\, \delta\alpha, \qquad \alpha(0, \eta) \geqslant 1 - \sqrt{\eta},$$

where c is an absolute constant.

For each η, iterate Item 3 and find $\tau_m < c_0 \alpha_{m-1}$, $1 \leqslant m \leqslant N$ where $\alpha_{m-1} = \alpha(\tau_{m-1})$, $\tau_0 = 0$, and $v_m = v(\tau_m)$ satisfies

$$\|v_m - v_\eta\| \leqslant \|v_{m-1} - u_\eta\| - \frac{1}{4} \alpha_{m-1} \|v_m - v_{m-1}\|$$

$$\|v_m - v_{m-1}\| \geqslant \frac{\tau_m}{2} \|K_{v_{m-1}}\|.$$

Combine and use the definition of α_m to obtain

$$\|v_m - u_\eta\| \leqslant \|v_{m-1} - u_\eta\| - \tfrac{1}{8}\alpha_{m-1}\tau_m \|K_{v_{m-1}}\| \leqslant \{1 - \tfrac{1}{8}\tau_m \alpha_{m-1}^2\} \; \|u_\eta - v_m\|$$

and

$$\|v_N - u_\eta\| \leqslant \|u_\eta - v\| \prod_{m=1}^{N} \{1 - \tfrac{1}{8}\tau_m \alpha_m^2\}. \tag{14}$$

For N large, we determine η and establish that

$$\prod_{m=1}^{N+1} \left\{1 - \frac{1}{8}\tau_m \alpha_{m-1}^2(\eta)\right\} \Big/ \sqrt{\eta} \leqslant \frac{\kappa}{2}.$$

Let $s = \sqrt{\eta}$ and $r_m = \tau_1 + \tau_2 + \cdots + \tau_m$. By construction $\alpha_m(s) = \alpha(r_m, s)$ where $\alpha_0(s) = \alpha(0, s) = \sqrt{1-s^2}$. The estimate is established by underestimating $\sum_{m=1}^{N} \tau_m \alpha^2(r_{m-1}, s)$. $\alpha(r, s)$ is underestimated by the solution $f(r, s)$ of the equation

$$\frac{\partial f}{\partial \tau} = -f + f^3 - s\,\delta f, \qquad f(0, s) = \sqrt{1-s^2},$$

where we have adjusted δ. $f(r, s)$ decreases in r and in s and

$$\lim_{r \to \infty} f(r, s) = 0 \qquad \text{and} \qquad \lim_{s \to 0} f(r, s) = 1.$$

Let real $0 < \gamma < 1$, $\tau_m = r_m - r_{m-1} = c_0^2 \{1/m\}^{1-\gamma}$, $1 \leqslant m \leqslant N+1$ and $c_0 \leqslant 1$ be the number determined in Item 3. For $f(r_{m-1}, s)$, $0 < s < 1$, use the properties of the function $f(r, s)$ to confirm the existence of $s = s'_{m-1}(\gamma)$, $1 \leqslant m \leqslant N$ for which

$$f^2(r_{m-1}, s'_{m-1}) \geqslant c_0^2 \left\{\frac{1}{m}\right\}^{\gamma}.$$

It follows that $\tau_m < c_0 f(r_{m-1}, s'_{m-1})$. To determine the sense in which s depends on N we need the following estimate. Multiply the equation for $f(r, s)$ by $f(r, s)$ and integrate from r' to r and use the fact that $f(r, s)$ is decreasing in r and confirm

$$\frac{1}{1-2(r-r')} f^2(r, s) \geqslant f^2(r', s) - \frac{2s\,\delta(r-r')}{1-2(r-r')}$$

provided that $r - r' < 1/2$. For $s < s'_{N-1}$, $r = r_N$ and $r' = r_{N-1}$, use the previous inequality, together with $f^2(r_{N-1}, s) > f^2(r_{N-1}, s'_{N-1})$ and previous estimates to establish, for $s = s_N = N^{2\gamma} c_N / 2\delta$, that

$$\tau_{N+1} f^2(\tau_N, s_N) \geqslant \frac{1}{4} c_0^4 \frac{1}{N+1}$$

and

$$\tau_{N+1} < c_0 f(r_N, s_N),$$

where c_N tends to $(1-2\gamma)/\gamma$ as N approaches infinity. Combine

$$f(r_{m-1}, s_N) > f(r_{m-1}, s'_{m-1}), \qquad m \leqslant N$$

with the previous estimates and then adjust δ to find

$$\prod_{m=1}^{N+1} \{1-\tfrac{1}{8}\tau_m \alpha_{m-1}^2(s_N)\}/s_N \leqslant N^{2\gamma-c'}(1-2\gamma)\,\gamma^{-1},$$

where $c' \leqslant 1$ is a small constant, and independent of v, N, and γ. Select γ and N independent of v to obtain the result.

Combine (14) with the estimate of the product to confirm

$$\|u_\eta - v_N\| \leqslant \frac{\kappa}{2}\sqrt{\eta}\,\|u_\eta - v\|. \tag{15}$$

By construction, the curve $v(\tau)$ corresponds to a curve $t(\tau) \in l_2$ and $v(\tau) = S(t(\tau))\,v$. Combine (15) with $\|u-u_\eta\|$ and obtain (12). By definition

$$\frac{(u_\eta - v(\tau),\, K_{v(\tau)})}{\|u_\eta - v(\tau)\|\;\|K_{v(\tau)}\|} = \|K_{v(\tau)}\|/\|u_\eta - v(\tau)\| < 1.$$

After adjusting δ apply (3) and obtain (13).

We combine and prove Lemma 2. Begin with v in M_{u_f} and $\|u-v\| \leqslant \delta$ where δ is selected as in Item 4 independently of κ. By the previous steps, find $t_m \in l_2$, $m \geqslant 1$ where $v_m = S(t_m)\,v_{m-1}$, $v_0 = v$ satisfies

$$\begin{aligned}\|S(t_m)\,v_{m-1} - u\| &\leqslant \left(\frac{\kappa}{2}\eta + \sqrt{1-\eta}\right)\|u - v_{m-1}\|,\\ &\leqslant \left(\frac{\kappa}{2}\eta + \sqrt{1-\eta}\right)^m \|u - v_1\|,\end{aligned}$$

and

$$c_1\,|t| \leqslant \frac{\kappa}{2}\eta\,\|u - v_{m-1}\| \leqslant \frac{\kappa}{2}\eta\left(\frac{\kappa}{2}\eta + \sqrt{1-\eta}\right)^{m-1}\|u-v\|.$$

It follows that v_m converges to u in L_2. Sum the estimate for $|t_m|$ and find

$$c_1\left|\sum_{m=1}^{N} t_m\right| \leqslant c_1 \sum_{m=1}^{N} |t_m| \leqslant \|u-v\|\,\frac{\kappa}{1-\kappa}$$

which confirms the convergence of $\sum_{m=1}^{\infty} t_m$ to t in l_2. By construction,

$$v_N = S\left(\sum_{m=1}^{N} t_m\right) v$$

is in H_m. It follows from the convergence of v_N to u in L_2 that $S(\sum_{m=1}^{N} t_m)\, v$ converges to $S(t)\, v$ in H_m and that $S(t)\, v=u$ satisfies

$$c_1\, |t| \leqslant \frac{\kappa}{1-\kappa}\, \|S(t)\, v - v\|.$$

This completes the proof.

Step 2. For f in a small enough neighborhood of $f_0 \in l$, there exists $\omega_m(f)$ in L_f that is a complete minimal sequence in l_2 and $\omega_m(f)$ converges to ω_m in l_2 as f tends to f_0. In addition, $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$ converges in l_2 uniformly in f and τ_m. The collection of all convergent sums $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$, $0 \leqslant \tau_m < 1$ with the identification of $\tau_m = 1$ and $\tau_m = 0$ is a compact subset of l_2.

Following [2], there exists ω_m in L_{f_0} that is a complete minimal sequence of l_2. There exist functions $\hat{\omega}_m$ on l_2 for which the l_2 product $\langle \omega_m, \hat{\omega}_n \rangle$ equals zero for $m \neq n$ and one if $m=n$. Select κ_m so that $\sum_{m=1}^{\infty} |\omega_m|\, |\hat{\omega}_m|\, \kappa_m$ is finite. By step one, there exists $\omega_m(f)$ for which $S(\omega_m(f))\, u_f = u_f$ and

$$c_1\, |\omega_m(f) - \omega_m|_{l_2} \leqslant \frac{\kappa_m}{1-\kappa_m}\, \|S(\omega_m)\, u_f - u_f\| \leqslant c_2\, \frac{\kappa_m}{1-\kappa_m}\, |\omega_m|_{l_2}, \qquad (16)$$

where c_1 and c_2 are absolute constants. Combine this inequality with the result of step one that $\|S(\omega_m)\, u_f - u_f\|$ approaches zero as f tends to f_0, to establish the result.

Since ω_m is a complete minimal sequence in l_2, $\omega \in l_2$ equals $\sum_{m=1}^{N} c_m \omega_m$ where $c_m = \langle \omega, \hat{\omega}_m \rangle$. Estimate

$$\left| \sum_{m=1}^{N} c_m(\omega_m(f) - \omega_m) \right|_{l_2}$$

by using (16), the fact that $\sum_{m=1}^{\infty} |\omega_m|\, |\hat{\omega}_m|\, \kappa_m$ is finite, and the convergence of $\omega_m(f)$ to ω_m in l_2 as f approaches f_0 to confirm

$$\left| \sum_{m=1}^{N} c_m(\omega_m(f) - \omega_m) \right|_{l_2} \leqslant \frac{1}{2} \left| \sum_{m=1}^{N} c_m \omega_m \right|_{l_2}. \qquad (17)$$

Use the proceeding inequality together with a small modification of the proof of a stability result due to Paley–Weiner [11] to prove that $\omega_m(f)$ is a complete minimal l_2 sequence. The convergence of $\sum_{m=1}^{\infty} \tau_m \omega_m$, $0 \leqslant \tau_m < 1$ in l_2 uniformly in τ_m and (17) establish that, in l_2, $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$ converges uniformly in τ_m and f. It follows that the collection of convergent series $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$, $0 \leqslant \tau_m < 1$ with the identification of $\tau_m = 1$ with $\tau_m = 0$ is a compact subset of l_2.

Step 3. In l_2, $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$ converges to $\sum_{m=1}^{\infty} \tau_m \omega_m$ uniformly in τ_m as f tends to f_0 in l. This follows directly from the convergence of $\omega_m(f)$ to ω_m in l_2 as f tends to f_0 in l and that in l_2, $\sum_{m=1}^{\infty} \tau_m \omega_m(f)$ converges uniformly in τ_m and f.

This completes the proof of the theorem.

4

We apply the result of this work and [2] to Korteweg–de Vries. Let u be a smooth periodic function. In this case, Gardner, Kruskal, and Muira [12, 13] constructed an infinite sequence of functionals $I_m(u)$ of the form

$$I_m(u) = \int_0^1 P_n(u)\, ds,$$

where $P_n(u) = a(D^n u)^2 + bD^n u D^{n-2} u + c(D_{n-1} u)^2 + dD^n u + eD^{n-1} u$, a is a constant and $(b, ..., e)$ are polynomials in $(D^j u)$, $j \leqslant n-1$. If $D^k u$ has weight $1 + k/2$ then $P_n(u)$ has weight $n+2$. The first three functionals are

$$\int_0^1 u\, dx, \qquad \int_0^1 u^2\, dx, \qquad \int_0^1 \left(\frac{u^3}{6} - \frac{u_x^2}{2}\right) dx.$$

$J = \partial/\partial x$ defines the Poisson bracket. $\{I_m(u), I_n(u)\}$ vanishes for all m and n and $I_m(u)$ are constant along solutions of

$$\frac{\partial u}{\partial t} = K_{I_m(u)} = \frac{\partial}{\partial x} G_{I_m(u)}, \qquad m \geqslant 1,$$

the generalized Korteweg–de Vries equation [13, 14].

The original Korteweg–de Vries equation is equivalent to

$$\frac{\partial u}{\partial t} = [Q, A],$$

where $A = 4D^3 - 3(uD + u_x)$ and $Q = -d^2/dx^2 + u$. The operator Q acting on smooth periodic functions has a discrete spectrum comprised of a simple periodic value λ_0 followed by alternately period two and period one pairs $\lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \cdots \to \infty$ of simple and for double eigenvalues with eigenfunctions $f_m(u)$. The function $y_1(x, \lambda)$, respectively $y_2(x, \lambda)$, is the solution of $Qy = \lambda y$ with $y_1(0, \lambda) = 1$, $dy_1(0, \lambda)/dx = 0$, respectively $y_2(0, \lambda) = 0$, $dy_2(0, \lambda)/dx = 1$. The periodic eigenvalues are the roots of the equation $y_1(1, \lambda) + dy_2(1, \lambda)/dx = 2$; period two eigenvalues are those of $y_1(1, \lambda) + dy_2(1, \lambda)/dx = -2$. The equation and initial condition confirm that

$y_1(x)\, dy_2(x)/dx - y_2(x)\, dy_1(x)/dx = 1$. Gardner, Kruskal, and Muira have shown that if $u(x, t)$ is a solution of the Korteweg–de Vries equation then the spectrum of the operator $Q(t)$ is independent of t. Lax [5] showed that the eigenvalues $\lambda_m(u)$ are constant under all the generalized KdV flows. Kruskal and Zabusky [4] observed that the functionals $I_m(u)$ are functions of the $\lambda_m(u)$. The portion of C_1^∞ for which all but finitely many eigenvalues are double is dense and open. We consider the general situation in which the spectrum of

$$-d^2/dx^2 + u_0$$

is simple. The eigenvalues satisfy

$$\lambda_{2m-1}, \lambda_{2m} = m^2\pi^2 + \int_0^1 u_0\, dx + 0(m^{-2})$$

uniformly in u_0 from bounded sets in H as m tends to infinity.

In what follows, the condition $\int_0^1 u\, dx = 0$ is imposed. Let $F_{2m}(u) = \lambda_{2m}(u) - m^2\pi^2$ and $F_{2m-1}(u) = \lambda_{2m-1}(u) - m^2\pi^2$ and consider the set M_{u_0} of smooth period one functions with the same spectrum as u_0. Denote by M the portion of C_1^∞ for which the spectrum of $-d^2/dx^2 + u$ is simple. The estimates of λ_m imply that $\sum_{m=1}^\infty (F_m(u))^2$ is finite and that the convergence is uniform in u from bounded sets in H_n. $F(u) = (F_1(u), F_2(u), ..., F_m(u), ...)$ maps M into l_2. Let $l = F(M)$. For each f in l, $F^{-1}(l)$ equals M_{u_f}. $I_m(u)$ is constant on M_{u_f} and Lax [4] showed that $\|u\|_\eta$ is controlled by $I_m(u)$, $m \geqslant 1$ for every n. This shows that M_{u_f} is compact in C_1^∞. The gradient of $F_m(u)$ is $d\lambda_m(u) = f_m^2(u)/\|f_m\|^2$. Lax [4] verified that the λ_m are in involution. Set $G_m(u) = f_m^2(u)$, $K_m(u) = m^{-1} D f_{2m}^2(u)$, and let N_u and T_u be the closure in L_2 of the span $G_m(u)$ and $K_m(u)$ respectively. For u in M_{u_f}, McKean and Trubowitz [1] established that $G_m(u)$ is a basis of N_u and that $K_m(u)$ is a basis for T_u and that $N_u \oplus T_u = L_2$. In particular, each G_u in N_u is uniquely expressible as $G_u = \sum_{m=1}^\infty t_m G_m(u)$ and $c_1 |t| \leqslant \|\sum_m t_m G_m(u)\| \leqslant c_2 |t|$, where c_1 and c_2 depend continuously on $u \in M$. In [2] we verified that $(dG_v K_v, K_v)/\|K_v\|^2 \leqslant c\, \|G_v\|$ where c is independent of v in M_{u_f}. By the result in [2], M_{u_f} is an infinite-dimensional torus. It is known that if u_0 is an element of M and $v \in H_n$ then $u_0 + \beta v$ for all β small enough is contained in M except for an countable number of values of β. This determines a local solution of

$$\frac{du}{d\tau} = R(u), \qquad u(0) = u_0$$

that is transverse to the level set M_{u_0}.

To complete the application, we verify that

$$\|dG_m(u)\,v\| \leqslant c_m\,\|v\|$$

for directions v transverse to M_u at u and c_m is independent of u in a small neighborhood of u_0 and v and is square summable. Let $\lambda = s^2$. $y_1(x, s)$ and $y_2(x, s)$ are expressed by

$$y_1(x, s) = \cos(sx) + \frac{1}{s}\int_0^x \sin(s(x-y))\,u(\eta)\,y_1(\eta, s)\,d\eta$$

and

$$y_2(x, s) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin(s(x-\eta))\,u(\eta)\,y_2(\eta, s)\,d\eta.$$

For s and s' bounded away from zero, use Gronwall's inequality to obtain

$$|y_1(x, s) - y_1(x, s')| \leqslant c\,|s-s'|$$
$$|y_2(x, s) - y_2(x, s')| \leqslant c\,|s-s'|,$$

where c is independent of s and s' and depends only on the bound $\max|u(x)|$. For s and s' bounded away from zero, the period one eigenfunctions $f(x, s) = y_2(1, s)\,y_1(x, s) + (1 - y_1(1, s))\,y_2(x, s)$ satisfy

$$|f(x, s) - f(x, s')| \leqslant c\,|s-s'|,$$

where c is an absolute constant. Period two eigenfunctions satisfy a similar estimate.

For a curve $u(\tau)$, $\tau \geqslant 0$ which is continuous in H and contained in M except for a countable number of values of τ with $u(0) = u$ in M and $\dot{u}(0) = v$ a direction transverse to M_u at u, we develop the bound

$$\|f_m(u)\,\dot{f}_m(u)\| \leqslant c_m\,\|v\|,$$

where c_m is independent of u and v and is square summable. First we consider period one functions f_{2m}, f_{2m-1} where m is even. In the present calculation $f_m = f_m(u(\tau))$, $\lambda_m(u(\tau))$ and $df_m/d\tau$ satisfies

$$\left\{-\frac{d^2}{d\tau} + u - \lambda_m\right\}\frac{df_m}{d\tau} = -f_m\frac{du}{d\tau} + f_m\frac{d\lambda_m}{d\tau}.$$

The right hand side of the previous identity is perpendicular to f_m. By completeness of period one eigenfunctions $f_l(u(\tau))$ and the proceeding differential expression, we find

$$\frac{df_m}{d\tau} = \sum_{l \neq m} \frac{(\dot{\lambda}_m f_m - \dot{u} f_m, f_l)}{(\lambda_l - \lambda_m) \|f_l\|^2} f_l.$$

For $l = 2m$, $2m-1$ use of the estimate for λ_l and λ_{2m} shows that, for j and m large, $|\lambda_{2l}(v) - \lambda_{2m}(v)| > c\, |l^2 - m^2|$ where c depends only on the bound for max $|v(x)|$. To estimate $df_{2m}/d\tau$ in L_2 we use $(f_{2m}, f_l) = 0$ to verify

$$\|f_{2m}\|^2 = \sum_l{}' \frac{(\dot{u}, f_{2m} f_l)^2}{(\lambda_l - \lambda_{2m})^2 \|f_l\|^2} + \frac{(\dot{u}, f_{2m} f_{2m-1})^2}{(\lambda_{2m} - \lambda_{2m-1})^2 \|f_{2m-1}\|^2},$$

where $'$ indicates that $l \neq 2m$, $2m-1$. For l' large enough, bound the terms $l > l'$ in the sum as follows

$$\begin{aligned} \sum_{l > l'}{}' \frac{(\dot{u}, f_{2m} f_l)^2}{(\lambda_l - \lambda_{2m})^2 \|f_l\|^2} &\leqslant \sum_{l > l'}{}' \frac{|f_{2m}|_\infty^2 \|v\|^2 \|f_l\|^2}{(\lambda_l - \lambda_{2m})^2 \|f_l\|^2} \\ &\leqslant \sum_{l > l'}{}' \frac{c\, \|v\|^2}{(l^2 - m^2)^2} \\ &\leqslant \frac{c\, \|v\|^2}{m^2}, \end{aligned}$$

where we have used the fact that $|f_{2m}|_\infty$ is bounded independently of m and that

$$\int_1^{m-1} \frac{dx}{(m_2 - x^2)^2} + \int_{m+1}^\infty \frac{dx}{(m^2 - x^2)^2}$$

is bounded by a fixed multiple of m^{-2}.

To estimate the remaining term, use $f_{xx} + uf = \lambda f$ and standard estimates to find for λ large that $f^2(x)/\|f\|^2 \leqslant c$ where c depends on the bound for u in H. Use $(f_{2m}, f_{2m-1}) = 0$ to write

$$(\dot{u}, f_{2m} f_{2m-1}) = \left(\frac{(f_{2m-1}^2, \dot{u})}{\|f_{2m-1}\|^2} f_{2m-1} - \dot{u} f_{2m-1}, f_{2m} - f_{2m-1} \right).$$

Then apply $|f_{2m}(u(\tau)) - f_{2m-1}(u(\tau))| \leqslant c\, |\sqrt{\lambda_{2m}} - \sqrt{\lambda_{2m-1}}|$ and estimates as follows

$$\frac{((f_{2m-1}^2, \dot{u})/(\|f_{2m-1}\|^2)\, f_{2m-1} - \dot{u} f_{2m-1}, f_{2m} - f_{2m-1})}{|\lambda_{2m-1} - \lambda_{2m}|\; \|f_{2m-1}\|}$$

$$\leqslant \frac{|f_{2m} - f_{2m-1}|_\infty}{|\lambda_{2m} - \lambda_{2m-1}|\; \|f_{2m-1}\|} \int_0^1 \left| \frac{(f_{2m-1}^2, \dot{u})}{\|f_{2m-1}\|^2} f_{2m-1} - \dot{u} f_{2m-1} \right| dx$$

$$\leqslant \frac{c\, |\sqrt{\lambda_{2m}} - \sqrt{\lambda_{2m-1}}|}{\|f_{2m-1}\|\; |\lambda_{2m} - \lambda_{2m-1}|} \left\{ \|\dot{u}\|\; \|f_{2m-1}\| + \|f_{2m-1}\| \; \frac{|(f_{2m-1}^2, \dot{u})|}{\|f_{2m}\|^2} \right\}$$

$$\leqslant \frac{c\, \|v\|}{\sqrt{\lambda_{2m}} + \sqrt{\lambda_{2m-1}}} \left\{ 1 + \frac{|f_{2m-1}|_\infty^2}{\|f_{2m-1}\|^2} \right\} \leqslant c\, \|v\| \, \frac{1}{m}.$$

Estimate the remaining terms in a similar manner. This leads to the result.

REFERENCES

1. H. McKean and E. Trubowitz, Hill's operator and Hyperelliptic function theory in the presence of infinitely many branch points, *Comm. Pure Appl. Math.* **29** (1976), 143–226.
2. M. Schwarz, Commuting flows and invariant tori: Korteweg–de Vries, *Adv. Math.* **89** (1991), 192–216.
3. P. Lax, A hamiltonian approach to the KdV and other equations, *in* "Nonlinear Evolution Equations" (Michael G. Grandall, Ed.), p. 207, Academic Press, New York, 1978.
4. P. Lax, Periodic solutions of the KdV equations, *Comm. Pure Appl. Math.* **28** (1975), 141–188.
5. P. Lax, Almost periodic solutions of the KdV equation, *SIAM Rev.* **18** (1976), 351–375.
6. H. McKean, Integrable systems and algebraic curves, *in* "Global Analysis" (M. Grmela and J. E. Marsden, Eds.), p. 83, Springer-Verlag, New York, 1979.
7. H. McKean and P. Van Moerbeke, The spectrum of Hill's equation, *Invent. Math.* **30** (1975), 217–274.
8. S. P. Novikov, The periodic problem for the Korteweg–de Vries equation, *Funktsional. Anal. i Prijochen* **3** (1974), 54–66.
9. B. A. Dubrovin and S. P. Novikov, A periodicity problem for the Korteweg–de Vries and Sturm–Liouville equations, *Soviet Math. Dokl.* **15** (1974), 1597–1601.
10. A. R. Its and V. B. Mateev, On hills operators with a finite number of lacunae, *Funct. Anal. Appl.* **9** (1975), 65–66.
11. R. Paley and N. Wiener, Fourier transforms in the complex domain, *Amer. Math. Soc. Colloq. Pub.* **19**, New York (1934).
12. C. S. Gardner, M. D. Kruskal, and R. M. Mira, Korteweg–de Vries and generalizations II existence of conservation laws and constants of motion, *J. Math. Phys.* **9** (1968), 1204–1209.
13. C. S. Gardner, Korteweg–de Vries equation and generalizations, IV, The Korteweg–de Vries equation as a hamiltonian system, *J. Math. Phys.* **12** (1971), 1548–1551.
14. M. Schwarz, The initial value problem for the sequence of generalized Korteweg–de Vries equations, *Adv. Math.* **54** (1984), 22–56.